The Physics of Lateral Stability[1]

This analysis focuses on the basic physics of lateral stability. We ask "Will a boat heeled over return to the vertical? If so, how long will it take? And what is the influence of waves on the answers?" There are other dimensions of stability that are not considered. For example: pitching and yawing are not considered; swamping due to low freeboard is ignored; the effect of irregular waves (steep faces, varying frequencies or direction) is left out.

Because much of the analysis is mathematical, a summary of the results should be useful to those less mathematically inclined. None of these results will be new insights to the seasoned mariner. The objective here is to explain *why* these well-known effects occur, not to simply state that they exist.

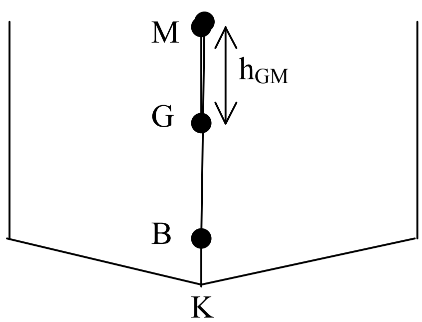
Summary

- **Stability is the result of two offsetting forces: gravity (downward) and buoyancy (upward). The result is a twisting force (torque) that restores stability; the torque increases with the heeling angle.**
- ***Rotational acceleration*, measuring the rapidity of restoration to neutrality, is proportional to the *righting arm* and inversely proportional to the square of the beam. Beamy boats, other things equal, are "tender," returning to neutrality more slowly than "stiff" boats.**
- **The "tenderness" of a vessel is measured by its *roll* period, which is directly proportional to the square of the beam, and inversely proportional to the square root of the *metacentric height* (which is directly proportional to the boat's righting arm).**
- **Roll period is independent of heeling angle: a slightly heeled vessel and a very heeled vessel will return to vertical in the same period, other things equal.**
- **Waves on the beam induce roll angles that come from two sources: the steepness of the wave at the boat's position, and the rate of recovery from previous roll. The latter depends on the size of prior roll angles and the rotational acceleration they induce.**
- **The roll angle induced by waves is at a maximum when waves arrive at the *resonant frequency*: higher or lower wave frequencies induce smaller roll angles than the resonant frequency.**
- **The maximum roll angle is directly proportional to wave height and inversely proportional to the coefficient of friction between boat and water. It is also smaller for larger boats.**

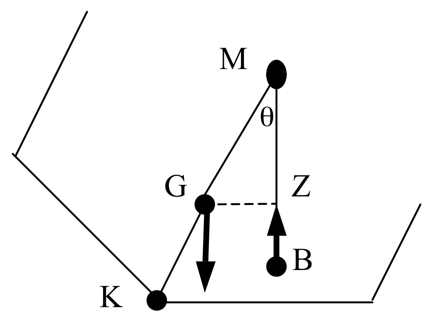
[1] The equations reported in these notes cannot be used without identifying the units of measurement. They are: meters for distance, seconds for time; kilograms for mass; radians for angular movements (rotations).

Fundamentals of Stability

Figure (1a) shows an unheeled vessel: the *Center of Buoyancy* (B), the *Center of Gravity (G)* and the *Metacenter* (M) all vertically above the Keel (K). This vertical line is called the *metacentric line*. Bouyancy exerts a vertical upward force (F_b) from point B; gravity exerts a downward force (F_g) from point G. The two forces are equal and offsetting ($F_b = -F_g$) so the vessel sits in the water with the weight of water displaced (its displacement) equal to the vessel's weight.



(1a)
Vessel Unheeled



(1b)
Vessel Heeled by θ°
Righting Arm (GZ) = GMsin(θ)

The center of gravity is the axis of rotation (the point around which the vessel rotates when it heels). The metacenter is (always) the point on the metacentric line directly above the center of buoyancy; the metacenter is calculated as KB + (I*/V), where KB is the distance from the keel to the center of buoyancy in figure (1a), V is the volume of water displaced (in cubic meters), and I* is the *second moment of area* (a measure of resistance to torsion) of the displaced water. The *metacentric height* is the distance between the metacenter and the center of gravity (GM, or h_{GM}).

Figure (1b) shows the vessel heeled to starboard at *heeling angle* θ. The center of buoyancy shifts rightward as the distribution of displaced water shifts. The vertical line BM shows the direction of the force of buoyancy, which is measured by the upward arrow at B. The original metacentric line is now rotated rightward at angle θ, pivoting on M.[2] The downward force of gravity is measured by the arrow at G.

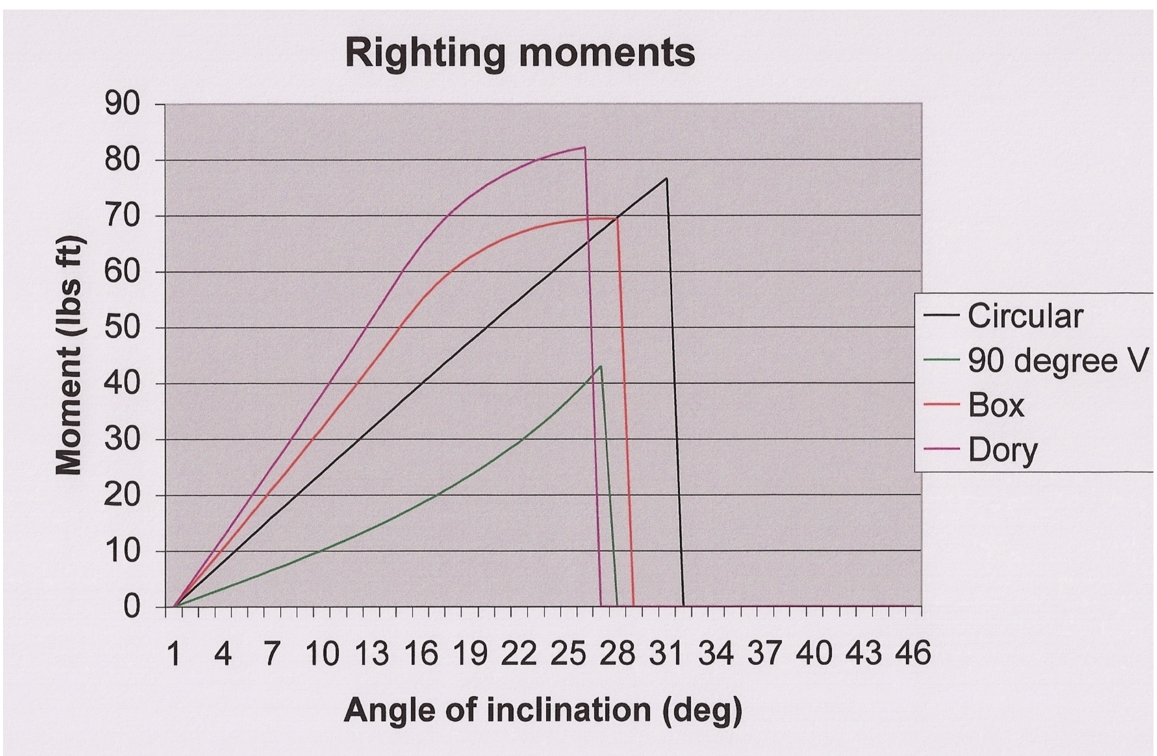
The forces of gravity and buoyancy always have the same magnitude but different signs: gravity's force is negative, bouyancy's is positive), so $F_b = - F_g$. For the heeled vessel these two forces act at different angles on the lever created by the righting arm.

[2] For small values of the heeling angle, the metacenter can be treated as fixed; as larger heeling angles are reached the metacenter shifts and the analysis is more complicated.

The result is a rotational force (twisting, torsion, torque) that restores a stable vessel to its original position, or capsizes an unstable vessel. The twisting force, or *torque*, is proportional to the *righting arm*'s length as measured by the distance GZ, the horizontal line from G to Z, where it makes a right angle with the line MB.

The righting arm is calculated as $h_{GM} \bullet \sin(\theta)$, where h_{GM} is the metacentric height. The righting arm increases as θ increases until θ reaches a point at which the righting arm vanishes. At a higher θ both the righting arm and the metacentric height become negative; the vessel is then unstable and capsizes because the torque exacerbates the heeling (the center of bouyancy is then to the left of the center of gravity).

The righting arm depends on the hull shape and the heeling angle. The chart below shows the righting arm length for several hull shapes, as a function of heeling angle. A box-shaped hull reaches the highest moment arm, but capsizes at the lowest heeling angle. A circular (rounded) hull capsizes at the highest heeling angle, but has a low righting moment at stable heeling angles, thus demonstrating the well-known fact that round-bottomed boats wallow but are more stable.



Torque and Stability

The analysis of the forces creating stability is equivalent to those governing the motion of a pendulum. To investigate this we use Figure (1c), an enlargement of Figure (1b), to examine torque.

The metacentric height (h_{GM}) is the length of the moment arm in a pendulum, with G being the point representing the mass at the end of the pendulum's arm. Torque—the force of rotation—is defined as the length of the moment arm times the force applied to it (F_T), i.e., $\tau_1 = rF_T$. In this case, $r = h_{GM}$ so $\tau_1 = h_{GM}F_T$.

The force applied can be seen using a force diagram to resolve the two forces in a pendulum (the gravitational force, directly downward, and the force traveling up the metacentric line from G to the pivot point at M). This diagram is the rectangle at point G. The angle at G is θ, the heeling angle, and the opposite side of that angle is $F_G\sin(\theta)$. The vector for torque force is pointed southeastward: torque pulls the center of gravity down and toward the center of buoyancy.Thus, $F_T = F_g\sin(\theta)$, so $\tau_1 = h_{GM}F_g\sin(\theta)$. Because $F_g = -mg$ we get the torque $\tau_1 = h_{GM}F_g\sin(\theta)$.[3]

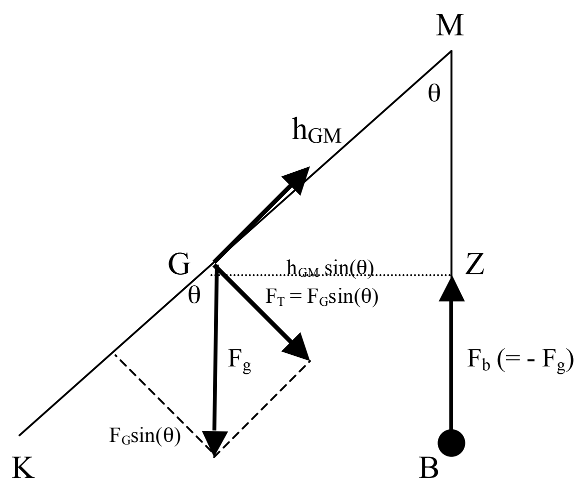


Figure (1c)

The gravitational force is $F_g = -mg$ so the torque created is

$$\text{(1a)} \qquad \tau_1 = -mg \cdot h_{GM} \cdot \sin(\theta)$$

[3] Note that this is identical to the torque measured in the more common approach, where torque is GZF_G or, because $GZ = h_{GM}\sin(\theta)$, torque is $h_{GM}F_G\sin(\theta)$.

Thus, torque is proportional to the metacentric height (and the righting arm) as well as to the vessel's mass, and it increases with the heeling angle. The torque is negative because it arises from the negative force of gravity.

An alternative of torque can be used to calculate the rotational acceleration created. This definition is that torque equals the *moment of inertia* (resistence to twisting, denoted by I) times the angular acceleration (denoted by α), that is, $\tau_2 = I\alpha$. The moment of inertia is defined as the boat's mass times the square of the *radius of gyration* (R_g)[4]. Thus,

$$(1b) \qquad \tau_2 = (m \bullet R_g^2)\alpha$$

Because the two definitions must give the same values for torque, we can derive the angular acceleration placed on the righting arm as

$$(1c) \qquad \alpha = -[g \bullet h_{GM} \bullet \sin(\theta)]/R_g^2$$

Rotational acceleration is directly proportional to the metacentric height (and to the length of the righting arm), inversely proportional to the square of the radius of gyration (it is greater for larger boats), and an increasing function of the heeling angle.[5] It is independent of the boat's mass. Note that as the boat rights itself, θ declines and rotational acceleration also declines. Footnote 6 describes the equation of motion for α.

For example, consider an open cockpit boat with an open shell hull. The radius of gyration is 1/2*beam. So $\alpha = -4g[h_{GM} \bullet \sin(\theta)]/beam^2$). For any given heeling angle, doubling the beam decreases the rotational acceleration by 75 percent.

Capsizing

As noted above, for small values of the heeling angle, the metacenter can be treated as fixed. But at sufficiently high heeling angles the metacenter shifts down the metacentric line. This fact is essential to vessel capsizing. The critical heeling angle is that for which the metacenter is at the center of gravity: the metacentric height and the righting arm vanish. A boat in this situation will neither right nor capsize. Higher heeling angles create a negative metacentric height and negative righting arm: the vessel capsizes: buoyancy reinforces rather than offsets the force of gravity.

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[4] R_g depends on the shape of the hull and the density structure of equipment, etc. within the hull. For a semicircular empty shell R_g is the outside radius (R). For a solid half-cylinder (which we assume for hull shape) $R_g = 0.71R$.

[5] The rotational acceleration is negative because it comes from the negative gravitational force. Thus, when we refer to g we mean the positive value: g = 9.8 meter/second^2.

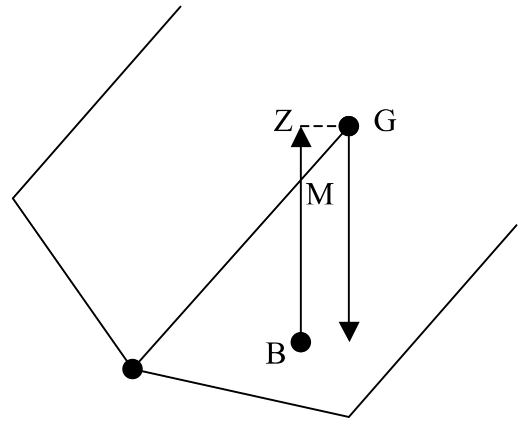

Figure (1c) Capsizing Vessel

Figure (1c) shows a capsizing vessel. The metacenter is below the center of gravity and the righting arm GZ is negative.The torque created is negative, the boat continues to roll, the heeling angle increases further, and the boat capsizes.

The Roll Period

How quickly does a vessel return to neutrality? The answer is measured by the roll period, defined as the theoretical time for the vessel with a given hull radius to rotate through a complete cycle from vertical to (say) θ° starboard, back to θ° to port, then back to vertical. We will see that roll period does not depend on the heeling angle: long rolls take as much time as short rolls! Roll time assumes no frictions or other damping forces; it also ignores exacerbating forces like waves.

Roll time is defined as $T= 2\pi/\omega$ where 2π is the radians in a complete cycle and ω is the angular frequency (radians per second). It turn out that the angular frequency is $\omega = \sqrt{[}(g\cdot h_{GM}/R_g^2] $ so
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$$\text{(2a)} \qquad T = 2\pi R_g/\sqrt{(g\cdot h_{GM})}$$

Roll time is directly proportional to the radius of gyration (longer for larger boats) and inversely proportional to the square root of the metacentric height (shorter for more stable boats).[6] As noted above, it is independent of the heeling angle.

An often-cited approximation to this is

$$\text{(2b)} \qquad T = beam*R_g/\sqrt{GM} = k*beam^2/\sqrt{GM}$$

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[6] The equality of torque definitions (τ_1 and τ_2) shown above yields the equation of motion $I\alpha = -\ mg\ h_{GM}\sin(\theta)$, where $\alpha = (d^2\theta/dt^2)$ is the angular acceleration and (for reference below) the angular velocity is $\omega \equiv d\theta/dt$. Also by definition, $I = mR_g^2$, so the equation of motion for the heeling angle is $(d^2\theta/dt^2) = -[gh_{GM}/R_g^2]\sin(\theta)$. This is a second-order differential equation. Using $\sin(\theta) = \theta$ as a linear approximation, the solution gives an angular velocity of $\omega = (1/R_g)\sqrt{(g\cdot h_{GM})}$. From which, noting that $T = 2\pi/\omega$, we get the roll period described in the text.

where R_g =k*beam and k = 0.4 < k < 0.55 depending on ship size and hull design. For a semicircular shell we have k = 0.5. A common value for large vessels (warships) is k = 0.40.

For example, consider a boat with k = .45, a 7 meter beam, and a 2 meter metacentric height. The roll time is approximately 16 seconds. This would be a "tender" (slow-rolling) boat.

Stability, Wave Action, and Resonance

Thus far it has been assumed that the vessel is in still water, that the neutral position of the boat is vertical, and that a single impulse has created the heel. A more realistic scenario is that continuous waves have created the roll. Introducing waves creates a distinction between the heeling angle and the roll angle: the heeling angle is the angle of rotation relative to the vertical position that will be achieved in flat water after torque has completed its work. The roll angle is the heeling angle plus the rotation introduced by variations in the wave face.

Introducing waves also raises the phenomenon of *reasonance*: as waves strike the beam at increasing frequency (going from, say, long ground swells to frequent and steep waves) the amplitude of the roll angle will increase. When wave frequency is just right, at the *resonant frequency*, the roll angle will reach a maximum. Higher wave frequencies will reduce roll angle from its maximum.

Consider a chain of waves coming on the beam in a regular sinusoidal pattern and creating the following equation for the height of the wave:

$$(3) \qquad w(t) = \mathbf{h} \cdot \cos(\omega t)$$

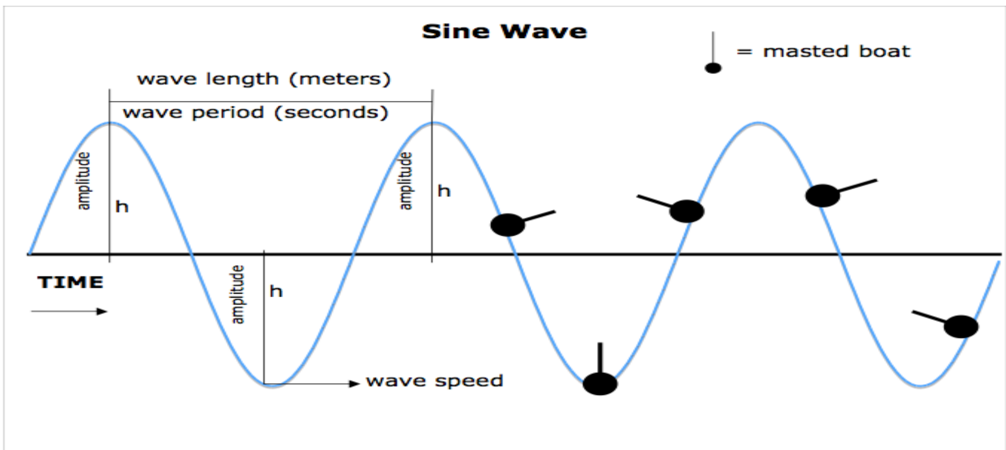
where **h** is the wave's amplitude (maximum crest or trough) in meters and ω is its angular velocity in seconds. T = 2π/ω is the wave period (time in seconds between crests), *f* = 1/T is the wave frequency (waves per second), λ is the wave length (distance in meters between crests), and υ = λ/T is the speed of the wave (in, say, meters per second).[7] Note that T here is not the T used above for roll period.

This wave train is represented in the figure below. The symbol  represents a masted boat in a vertical position. Imagine the boat as a chain of waves comes on the port beam. Suppose that the departure of the boat from the vertical represents rolling due entirely to wave action. Then the boat will move vertically with the waves, always remaining perpendicular to the face of the wave on which it rests. An additional source of

[7] It is typically assumed that υ = √gλ/2π in deep water. A wave with 3 meter length travels at 2.16 meters per second (7.8 kilometers per hour).

roll (not shown) is the heeling angle induced as the boat recovers from prior wave motion.

This representation of roll due to waves is highly stylized. It ignores the steepness of non-sine waves as well as the influence of breaking waves and of shifting payloads. It also neglects the phenomenon of waves arriving from different directions in a storm. Even so, it is a useful approach to the fundamental issues raised by wave action.



When a wave train comes on the beam it will affect the roll angle experienced by the boat. All three figures show a boat on the face of a wave with roughly a 45° angle. Figure (2a) shows the position of the boat if it has completely righted itself from any previous wave action; the boat is perpendicular to the wave face. Subsequently, the boat's roll angle will change for two reasons: first, the wave face angle will change; second, the boat will attempt to right itself from the position shown in figure (2a).

Figure (2b) shows the same boat but it is now vertical because it has not yet responded to the wave face. righting itself from a previous position on the wave face. This boat is "tender," having a low roll time relative to the more "stiff" boat in Figure (2c). The boat in (2c) is in a later stage of righting.

Figure (2a)

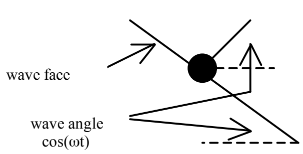


Figure (2b)

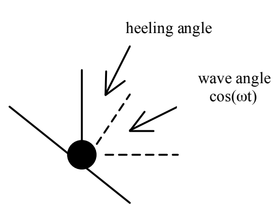
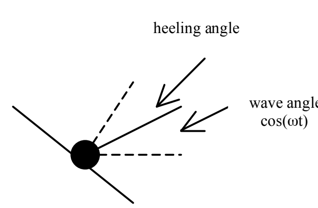


Figure (2c)



Thus, the roll angle is created by both the wave face and by the boat's roll time. Denny (2008)[8] shows, with some simplifying assumptions, that the roll angle from both sine waves and heeling is

$$(4) \qquad \theta(t) = \mathbf{a}\cdot\cos(\phi + \omega t)$$

in which the amplitude of the roll angle is **a** and ϕ, the phase angle, is the time delay between the wave face change and the boat's righting torque.

The roll angle's amplitude is a function of wave and boat characteristics that are embedded in **a**. Denny shows that the equation of motion for the roll angle, θ, is a second order differential equation[9] and that the amplitude of the roll angle is

$$(5) \qquad \mathbf{a} = \Omega_1^2/\sqrt{[\Omega_0^2 - \omega^2)^2 + \mathbf{b}^2\omega^2]}$$

where $\omega = \sqrt{[2\pi g/\lambda]}$, $\Omega_0^2 = gh_{CG}/R_g^2$, and $\Omega_1^2 = \sqrt{[(2\pi h/\lambda)(gh_{CB}/R_g^2)]}$.

The parameters h_{CG} and h_{CB} are distances from the center of the deck (flat side of a half-cylinder) to the centers of gravity and buoyancy, respectively.

Knowing the parameters allows calculation of the roll angle as waves arrive and depart. In addition, if waves arrive at the resonant frequency, the roll angle will be at a maximum. The resonant frequency of waves is that for which $d\mathbf{a}/d\lambda = 0$. From equation (5) we find that this requires angular velocity shown in (6a), and it occurs when the wave length is as shown in (6b):

$$(6a) \qquad \omega = \sqrt{[gh_{CG}/R_g^2]}$$

$$(6b) \qquad \lambda = 2\pi R_g^2/h_{CG}$$

The maximum roll angle amplitude is

$$(7) \qquad \mathbf{a}_{max} = \sqrt{[gh_{CG}h_{CB}h/bR_g^3]}$$

Thus, when resonance occurs, that is, when the angular velocity of the wave cycle is wave length is precisely the value shown in (7a) and the wavelength is the value in (7b), the maximum roll angle occurs. This roll angle increases with wave height, decreases with boat friction and with the radius of gyration. The boater seeking comfort

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[8] Denny's analysis (pp. 161-163, notes 5, 6 on p. 238) isbased on a number of simplifying assumptions, among them: The hull is assumed to be a solid half-cylinder; wave velocity is $v = \sqrt{g\lambda/2\pi}$ (the deep water value).

[9] The equation is $d^2\theta/dt^2 + \mathbf{b}\bullet(d\theta/dt) + \Omega_0^2 = \Omega_1^2\bullet\cos(\omega t)$. The slope of the wave face is $\cos(\omega t)$. This is the equation for forced-damped harmonic oscillation.

should look for larger boat, lower seas, and more boat friction (keel, stabilizers, flopper-stoppers, etc.

References

Denny, Mark. *Float Your Boat! The Evolution and Science of Sailing*, Johns Hopkins University Press, Baltimore, 2009.

Kimball, John. *The Physics of Sailing*, CRC Press, New York, 2010